and

$$PB = \frac{(x-10)(100-x)}{x}.$$

Again

$$\overline{PB}^2 = \overline{PD}^2 + \overline{BD}^2 \text{ or } \overline{PB}^2 = 10^2 + (x-10)^2 \text{ or } x^2 - 20x + 200.$$

Therefore

$$PB = \sqrt{x^2 - 20x + 200},$$

and

$$\frac{(x-10)(100-x)}{x} = \sqrt{x^2 - 20x + 200}.$$

Reducing, we get the cubic,

$$2x^3 - 139x^2 + 2200x - 10000 = 0.$$

Applying Descartes' rule of signs we find that all of the roots of the above cubic are positive. Applying Sturm's theorem we find that the roots are situated as follows: one root between 9 and 10; one root between 11 and 12; and one root between 49 and 50.

The root between 9 and 10 must be discarded, as it is impossible in our problem. Applying Horner's method for incommensurable roots, we find the roots to be 11.282 and 49.212 correct to three decimal places. Therefore $BC = 11.282$ or 49.212. Hence, the hypotenuse can be *88.718* ft. or *50.788* ft.

Also solved by HERBERT N. CARLETON, J. W. CLAWSON, HORACE OLSON, M. HELEN KELLEY, C. E. FLANAGAN, W. E. WHITFORD, G. H. HARTWELL, and NATHAN ALTSHILLER.

465. Proposed by ROGER A. JOHNSON, Western Reserve University.

Let C be a fixed circle, A a point outside it. Let AT and AT' be the tangents from A to the circle, touching the latter at T and T'. Let two secants be drawn through A, cutting the circle at P, Q and R, S respectively. Let PR and QS meet at X, PS and QR meet at Y. Prove by elementary methods that for all positions of the secants, X and Y lie on the line TT'.

SOLUTION BY J. W. CLAWSON, Collegeville, Pa.

I. $PQSR$ is a quadrangle, having X, Y, A for its diagonal points. Hence $X(ARYS)$ is a harmonic pencil. Then, if XY cuts ARS at Z, $(ARZS)$ is a harmonic range.

Now TT' is the polar of A with respect to the circle whose center is C. Hence, if TT' cuts ARS at Z', $(ARZ'S)$ is a harmonic range.

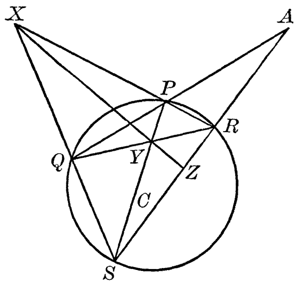

FIG. 1.

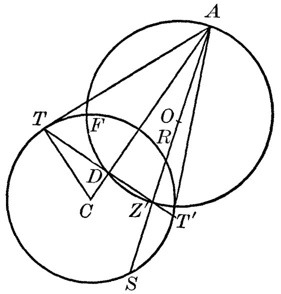

FIG. 2.

Therefore, the points Z and Z' coincide and XY and TT' cut the line ARS at the same point. Similarly, it can be shown that XY and TT' cut the line APQ at the same point. Hence the lines XY and TT' coincide.

[The exercise follows at once from the theorem: If a system of conics circumscribe a given quadrangle, the diagonal point triangle is a self-conjugate triangle w. r. t. each conic of the system. (Durell, Plane Geometry for Advanced Students, Part ii, p. 110.)]

If the above proof is not considered "elementary," a more detailed proof, avoiding the use of the terms "harmonic range" and "polar" is as follows:

II. Extend XY to meet ARS at Z.

Then, since XZ, SP, RQ are concurrent at Y, by Ceva's Theorem,

$$\frac{SZ}{ZR}\cdot\frac{RP}{PX}\cdot\frac{XQ}{QS}=1.$$

Again, since APQ is a transversal cutting the sides of the triangle XSR, by Menelaus' Theorem,

$$\frac{SA}{AR}\cdot\frac{RP}{PX}\cdot\frac{XQ}{QS}=-1.$$

Comparing these results, we see that

$$\frac{SZ}{ZR}=-\frac{SA}{AR}.$$

Again, let TT' cut SR at Z'. From A draw AD perpendicular to TT'. On AZ' as diameter draw a circle. This circle passes through D. Bisect AZ' at O. Let one of the points of intersection of the two circles be F.

Now, $\triangle\!\!\!\triangle$ ACT, TCD are similar. Hence $CT^2=CD\cdot CA$, *i. e.*, $CF^2=CD\cdot CA$. Therefore, CF is a tangent to circle O. Hence, $\angle CFO$ is a right angle. Hence, OF is a tangent to circle C. Hence, $OF^2=OR\cdot OS$, *i. e.*, $OZ'^2=OR\cdot OS$, *i. e.*,

$$\frac{OS}{OZ'}=\frac{OZ'}{OR},$$

i. e.,

$$\frac{OS-OZ'}{OS+OZ'}=\frac{OZ'-OR}{OZ'+OR},$$

i. e.,

$$\frac{Z'S}{AS}=\frac{RZ'}{AR}.$$

Hence,

$$\frac{SZ'}{Z'R}=-\frac{SA}{AR}.$$

Hence, the points Z and Z' coincide. Therefore, the lines XY and TT' intersect ARS at the same point. Similarly it can be shown that XY and TT' intersect APQ at the same point. Hence XY and TT' coincide.

Also solved by N. P. PANDYA.

CALCULUS.

378. Proposed by ELBERT H. CLARKE, Purdue University.

The area of the curved surface generated by the revolution about OX of the portion of the curve $y=x^n$ which extends from the origin to the point (1, 1) is given by the formula

$$A=2\pi\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx.$$

Our geometric intuition would tell us that the limit of this area as n becomes infinite is π. Give a strict analytic proof that

$$\lim_{n\doteq\infty}\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx=\tfrac{1}{2}.$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We give the proof by writing the above integral as the sum of two and showing that the limit of one is zero, of the other, $\frac{1}{2}$. Let k be any value between 0 and 1, *e. g.*, .9. Then

$$\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx=\int_0^k x^n\sqrt{1+n^2x^{2n-2}}\cdot dx+\int_k^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx.$$